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ON OPTIMAL MATCHINGS

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Given n random blue and n random red points on the unit square, the transportation cost between them is tipically $\sqrt{n \log n}$.

Given two sets of points $X_1, \ldots, X_n$, and $Y_1, \ldots, Y_n$, the transportation cost is defined as

$$T_n = \min_{\pi} \sum_{i=1}^{n} d(X_{\pi(i)}, Y_i),$$

where d stands for distance, and the min is taken over all permutations π of the integers $1, \ldots, n$. The purpose of the paper is to investigate the transportation cost for random input X, Y.

Assume that the points $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are chosen at random, independently of each other, with a uniform distribution on the unit square Q.

Theorem

$$C_1 (n \log n)^{1/2} < T_n < C_2 (n \log n)^{1/2} \quad (1)$$

with probability $1-o(1)$.

A slightly weaker upper bound

$$T_n = O(n^{1/2} \log n)$$

has been achieved by Karp.

In one sense the two dimensional case is the most interesting case, for in one dimension T_n can be determined explicitely and thus its magnitude is easily seen, while in dimension $r \geqq 3$ T_n is as small as $n^{1-1/r}$, which is a trivial lower bound for typical T_n, since nearest neighbours are at a distance $n^{-1/r}$.

(Observe that in one dimension the optimal matching is the merging of the two ordered samples. In case $r \geqq 3$ the final transport cost is resulted by a convergent sum which fact explains the absence of an extra $(\log n)^c$ term.)

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The Theorem generalizes to

$$T_n(p) = \min_{\pi} \sum_{i=1}^{n} d^p(X_{\pi(i)}, Y_i)$$

leading to the estimate $n(\log n/n)^{p/2}$, reflecting that a typical blue point moves a distance $(\log n/n)^{1/2}$ to the coupled red point.

Proof. First we will prove the lower part of (1). Following the method of Ford and Fulkerson, we construct a Lipschitzian function f on Q such that

$$A(f) = \sum_{i=1}^{n} f(X_i) - \sum_{i=1}^{n} f(Y_i) \geqq LC_1(n \cdot \log n)^{1/2}, \tag{2}$$

where L is the Lipschitzian parameter of f i.e.

$$|f(x)-f(y)| \leqq L\,d(x,y), \quad x, y \in Q. \tag{3}$$

Then, for arbitrary permutation π,

$$T_n = \sum_{i=1}^{n} d(X_{\pi(i)}, Y_i) \geqq \frac{1}{L} \sum_{i=1}^{n} |f(X_{\pi(i)}) - f(Y_i)| \geqq C_1 (n \log n)^{1/2}.$$

Roughly saying the construction goes as follows. Let $m=0.1 \log n$. For $i=1, 2, \dots, m$ let Q_{ij}, $j=1, 2, \dots, 4^i$ denote the subsquares of Q obtained by dividing both coordinates into 2^i equal parts. Set

$$X(Q_{ij}) = \sum_k I(X_k, Q_{ij}), \quad Y(Q_{ij}) = \sum_k I(Y_k, Q_{ij}), \quad V(Q_{ij}) = X(Q_{ij}) - Y(Q_{ij}), \tag{4}$$

where $I(x, A)=1$ for $x \in A$ and $I(x, A)=0$ otherwise. Then for the function $f(u) = \sum_{ij} V(Q_{ij})\, I(u, Q_{ij})$

$$A(f) = \sum_{ij} V^2(Q_{ij}) \sim Cn \log n. \tag{5}$$

(Here and from now on C denotes absolute constans with values varying from formula to formula.) For each fixed index i in (5) the summation on j results terms of magnitude Cn while i runs up to $m=0.1 \log n$.
Unfortunately, this function f is not Lipschitzian. That is why we redefine V and f as follows. Let $D(x, A)$ denote the distance of the point x and the set A, and $D(A)$ the diameter of A. Now we set

$$V(Q_{ij}) = \sum_k \frac{D(X_k, \bar{Q}_{ij}) - D(Y_k, \bar{Q}_{ij})}{D(Q_{ij})}, \tag{6}$$

and

$$f(u) = \sum_{i,j} \frac{V(Q_{ij})}{D(Q_{ij})} D(u, \bar{Q}_{ij}),$$

where bar indicates complementary set. This function is Lipschitzian already, but with a large constant. The Lipschitzian constant of the pyramidal function $D(u, \bar{Q}_{ij})$ is 1, the value of the coefficient $V(Q_{ij})/D(Q_{ij})$ is approximately normal with variance about n, and for any $u \in Q$ the number of non-void terms in the sum is $m=$

$0.1 \log n$. The sum of these non-void terms is typically $O((n \log n)^{1/2})$, but at some points u the effects of the large deviations add up to $n^{1/2} \log n$.

For cutting down the Lipschitzian constant from $n^{1/2} \log n$ to $(n \log n)^{1/2}$ an appropriate stopping rule will be introduced.

Let Q_i^u denote the Q_{ij} that contains u and

$$f_i(u) = \frac{V(Q_i^u)}{D(Q_i^u)} D(u, \bar{Q}_i^u).$$

Then

$$f_i'(u) = \frac{V(Q_i^u)}{D(Q_i^u)} D'(u, \bar{Q}_i^u),$$

and the derivate $D'(u, \bar{D}_i^u)$ exists in the interiors of the four triangles into which the diagonals split the square Q_i^u. The values of this derivative are the four vectors $(\pm 1, 0)$, $(0, \pm 1)$. Let the stopping time $t(u)$ be defined as the largest $t \leqq m$ such taht

$$\sup_{v \in Q_t^u} \max_{1 \leqq k \leqq t} \left| \sum_{i=1}^{k} f_i'(v) \right| \leqq (Cn \log n)^{1/2}. \tag{7}$$

Now the final version of the function f is

$$f(u) = \sum_{i=1}^{t(u)} f_i(u). \tag{8}$$

This function f is continuous and piecewise differentiable, hence (7) implies that f is Lipschitzian with constant $L = \sqrt{Cn \log n}$.

Now we are going to show that for this f we still have $A(f) \geqq Cn \log n$ with probability $1 - o(1)$. Our stopping rule is such that the inequality $t(u) \geqq i$ implies $t(v) \geqq i$ for all $v \in Q_i^u$. Let U_{ij} be equal to 1 if $t(u) \geqq i$ on Q_{ij}, and zero otherwise. Then

$$A(f) = \sum_{i,j} U_{ij} V^2(Q_{ij}).$$

The sequence of functions τ_i defined by $\tau_i(u) = 1$ if $t(u) \geqq i$, and $\tau_i(u) = 0$ otherwise, is monotone decreasing, hence so is the sequence of their integrals

$$U_i = \int_Q \tau_i(u)\, du = 4^{-i} \sum_j U_{ij}.$$

The main point is that

$$U_m \geqq 1/2 \tag{9}$$

with probability $1 - o(1)$. This implies that $A(f) \geqq Cn \log n$, because with probability $1 - o(1)$ we have

$$\sum_{j=1}^{4^i} \varepsilon_j V^2(Q_{ij}) \geqq Cn \tag{10}$$

for any i and any choice of numbers $\varepsilon_j = 0, 1$, such that

$$\sum_{j=1}^{4^i} \varepsilon_j \geqq \frac{1}{2} 4^i$$

if only C is small enough.

For proving (9) and (10) the following embedding theorem can be used ([3], [5]): Set

$$G(u) = \sum_{k=1}^{n} I(X_k, R^u), \quad u \in Q,$$

where R^u denotes the rectangle connecting the origin with the point u. Then there is a two-dimensional Wiener process $W(u)$ such that the Brownian bridge

$$B(u) = W(u) - \lambda(R^u)W(1)$$

(where $\lambda(A)$ stands for the area of the set A) approximates $G(u)$ in the following sense

$$(11) \qquad P\big(\sup_{u\in Q} \left|G(u) - (n\lambda(R^n) + n^{1/2}B(n))\right| \geqq \log n(C\log n + x)\big) \leqq Ke^{-\alpha x},$$

C, K, α are positive absolute constants. Hence the process $V(Q_{ij})$ is approximated by the corresponding Gaussian process $W(Q_{ij})$ as near as $C\log^2 n$ with probability $1-o(1)$. For the Gaussian process $W(Q_{ij})$ (9) and (10) are easily seen.

For defining the Wiener process $W(Q_{ij})$ it is enough to postulate that for fixed index i the random variables $W(Q_{ij})$ are independent Gaussian variables, and the set-function $W(\,.\,)$ is additive.

This implies the validity of (10) for the process $V(Q_{ij})$ since the number of terms in (10) is $4^i \leqq 4^m \ll n/\log^2 n$. Similarly, (9) is implied for $V(Q_{ij})$ with a possibly smaller C in definition (7). Now we turn to proving the upper part of (1). We are going to assign to every point X_i a rectangle R_i^x of area $1/n$ such that these rectangles form a partition of Q, and the transportation cost

$$S_n^X = \sum_{i=1}^{n} \int_{R_i^X} d(x_i, u)\, du$$

satisfies

$$S_n^X < C(n\log n)^{1/2}$$

with probability $1-o(1)$. A similar construction for the points Y_i and the triangle inequality

$$T_n \leqq S_n^X + S_n^Y$$

yield the upper part of (1). (Indeed, spreading the points X_i uniformly on the corresponding R_i^x and spreading the points Y_i uniformly on the corresponding R_i^Y lead to the same uniform measure on Q, and thus they together determine a transportation from the points X_i to the points Y_i. The best transportation however is always a matching.)

Now given a set of points $X_1, \dots, X_n$ in Q, we construct a monotone sequence of partitions $\{R_{i1}, R_{i2}, \dots\}$ $i=1, 2, \dots$ such that the R_{ij} are (possibly degenerated) rectangles and

$$(12) \qquad \lambda(R_{ij}) = \frac{1}{n} X(Q_{ij}), \quad i = 1, 2, \dots, \quad j = 1, 2, \dots, 4^i,$$

(see (4)). Let us number the four quadrants Q_{11}, Q_{12}, Q_{13}, Q_{14} of Q as follows:

$\begin{pmatrix}1&2\\3&4\end{pmatrix}$. We push the horizontal bisector of Q for getting two halves of areas

$$\frac{1}{n}(X(Q_{11})+X(Q_{12})) \quad \text{and} \quad \frac{1}{n}(X(Q_{13})+X(Q_{14})).$$

In the upper half we push the vertical bisector to obtain the rectangles R_{11} and R_{12} with areas given in (12), and then do the same in the lower half. Note that the new vertical walls are not necessarily collinear. We can imagine the unit square Q as being made of a flexible membrane, and when pushing the walls the interiors of the quadrants transform in an affine way. In the next step we draw the bisectors within each quadrants R_{11}, R_{12}, R_{13}, R_{14} and then pushing the walls just as above we get the subrectangles $R_{2,1} \dots, R_{2,16}$ satisfying (12). We follow the same refining for all $i \leqq M = \log n$. For a typical random sample X_i, every cell Q_{Mj} contains at most one sample point and thus every non-degenerate R_{Mj} is of area $1/n$.

Now we estimate the transportation cost. The height of R_{k1} is

$$h_{k1} = \prod_{i=1}^{k} \frac{X(Q_{i1})+X(Q_{i2})}{X(Q_{i1})+X(Q_{i2})+X(Q_{i3})+X(Q_{i4})} = \prod_{i=1}^{k} \frac{X(Q_{i1})+X(Q_{i2})}{X(Q_{i-1,1})},$$

(where $Q_{01}=Q$), and the width of R_{k1} is

$$w_{k1} = \prod_{i=1}^{k} \frac{X(Q_{i1})}{X(Q_{i1})+X(Q_{i2})}.$$

The numbers h_{kj} and w_{kj} are of similar form. It is more complicated to follow the movement of the points of Q. Let u be an arbitrary point in Q. If Q_{ij} contains u, and w and h denote the distances of u from the nearest vertical and horizontal walls of Q_{ij}, then after the i-th step the image u_i of u in R_{ij} will have corresponding distances $w2^i w_{ij}$, $h2^i h_{ij}$. Hence in the $(i+1)$-st step the vertical shift of u_i will be

$$h2^{i+1}h_{i+1,j_1} - h2^i h_{ij} = h2^i h_{ij} \frac{X(Q_{i+1,j_1})+X(Q_{i+1,j_2})-X(Q_{i+1,j_3})-X(Q_{i+1,j_4})}{X(Q_{i+1,j_1})+X(Q_{i+1,j_2})+X(Q_{i+1,j_3})+X(Q_{i+1,j_4})}$$

and the horizontal shift will be

$$w2^{i+1}w_{i+1,j_1} - w2^i w_{ij} = w2^i w_{ij} \frac{X(Q_{i+1,j_1})-X(Q_{i+1,j_2})}{X(Q_{i+1,j_1})+X(Q_{i+1,j_2})},$$

where Q_{i+1,j_1}, Q_{i+1,j_2}, Q_{i+1,j_3}, Q_{i+1,j_4} are the four quadrants of Q_{ij} numbered as before and for simplicity of notation we assumed that u lies in Q_{i+1,j_1}. The magnitude of both w and w_{ij} is 2^{-i}, that of $X(Q_{i+1,j_1})+X(Q_{i+1,j_2})$ is $n4^{-i}$, and that of $X(Q_{i+1,j_1})-X(Q_{i+1,j_2})$ is $n^{1/2}2^{-i}$. Hence the magnitude of the horizontal shift is $n^{-1/2}$. The same holds for the vertical shift. These shifts are symmetrically distributed random variables and the shifts of a given point in different steps are nearly independent, thus it can be expected that their sum is of order $M^{1/2}n^{-1/2}$. For proving this latter statement one can again use the embedding (11) as long as $n2^{-i} \gg \log n$ and estimating the effect of the remaining $\log\log n$ steps simply by the sum of absolute values of the shifts, i.e. by $o(n^{-1/2}\log\log n)$.

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